BUCKLING OF A CYLINDRICAL SHELL LOADED BY A PRE-TENSIONED FILAMENT WINDING

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INTRODUCTION

Designs have appeared recently in launch-vehicle technology in which circular cylindrical shells are wrapped with pre-tensioned filaments. An example of such a design is a cylinder which is wrapped circumferentially to help carry hoop stress from internal pressure. Another example is a cylindrical cryogenic tank upon which insulation is held in place by pre-tensioned filaments to protect against aerodynamic heating. One question that arises in the design of such configurations is: What is the largest filament tension that can be allowed before buckling of the shell would occur? It is believed that the theoretical analysis presented in this note is a contribution toward answering this question.

An idealized model of the configuration is studied in which material compressibility in the thickness direction is represented by many closely spaced, independent radial springs located between the shell and the filaments. For the case of insulation secured to the shell by filament winding, the springs represent essentially the thickness compressibility of the insulation. For the case of filaments wrapped directly on the shell, the springs represent some effective thickness compressibility of the shell and filaments. A simple, closed-form solution is obtained for the idealized model. Results are plotted in dimensionless form so that the buckling load (the critical tension in the filaments or

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the critical compression in the shell) can be obtained directly for wide ranges of the geometrical and stiffness parameters of the model.

NOMENCLATURE

D	plate flexural stiffness, $Et^3/12(1 - \mu^2)$
E	Young's modulus
k	elastic spring constant
k_y	buckling coefficient, $NL^2/D\pi^2$
K	spring stiffness parameter, $\frac{kL^4}{\pi^4 D}$
L	length of cylinder
n	number of waves in circumferential direction
N	applied circumferential stress resultant
N_x, N_y, N_{xy}	in-plane stress resultants
r	radius of cylindrical shell
t	thickness of cylindrical shell
u,v,w	displacements in the x-, y-, and radial directions, respectively
x,y	axial and circumferential coordinates, respectively
Z	curvature parameter, $\frac{L^2}{rt}\sqrt{1 - \mu^2}$
$\epsilon_x, \epsilon_y, \gamma_{xy}$	in-plane strains
μ	Poisson's ratio

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

∇^{-4}	inverse operator defined by equation $\nabla^{-4}(\nabla^4 f) = \nabla^4(\nabla^{-4} f) = f$
o	subscript referring to filament winding

When the subscripts x and y follow a comma, they indicate partial differentiation with respect to x and y, respectively.

ANALYSIS

The buckling of an elastic isotropic, homogeneous, circular cylindrical shell loaded through many closely spaced radial springs by pre-tensioned circumferentially wrapped filaments is analyzed (see fig. 1). The filament winding is considered to have no bending stiffness. The difference between the radius of the shell and of the filaments is assumed to be negligible so that the hoop tension per unit length in the filaments, N causes an equal magnitude of hoop compression, $-N$, in the shell. The filaments are permitted to slide freely relative to the cylinder and relative to each other; i.e., no bonding agent is applied to the filaments or the cylinder. Thus the hoop tension load per unit length in the filament winding is simply the number of filaments per unit length times the load in each filament.

For changes in stress resultants and displacements which occur during buckling the equilibrium equations are:

For the shell

$$N_{x,x} + N_{xy,y} = 0 \tag{1}$$

$$N_{y,y} + N_{xy,x} = 0 \tag{2}$$

$$D\nabla^4 w + \frac{N_y}{r} + Nw_{,yy} + k(w - w_o) = 0 \tag{3}$$

For the filament winding

$$\frac{dN_{yo}}{dy} = 0 \tag{4}$$

$$\frac{N_{yo}}{r} - N\frac{d^2w_o}{dy^2} - k(w - w_o) = 0 \tag{5}$$

The stress resultant-strain relations are:

For the shell

$$\left.\begin{aligned} N_x &= \frac{Et}{1-\mu^2}\left(\epsilon_x + \mu\epsilon_y\right) \\ N_y &= \frac{Et}{1-\mu^2}\left(\epsilon_y + \mu\epsilon_x\right) \\ N_{xy} &= \frac{Et}{2(1+\mu)}\gamma_{xy} \end{aligned}\right\} \tag{6}$$

For the filament winding

$$N_{yo} = E_o t_o \epsilon_{yo} \tag{7}$$

The strain-displacement relations are:

For the shell

$$\left.\begin{aligned} \epsilon_x &= u_{,x} \\ \epsilon_y &= v_{,y} + \frac{w}{r} \\ \gamma_{xy} &= u_{,y} + v_{,x} \end{aligned}\right\} \tag{8}$$

For the filament winding

$$\epsilon_{yo} = \frac{dv_o}{dy} + \frac{w_o}{r} \tag{9}$$

In a procedure similar to that used in reference 1, equations (1), (2), (3), (6), and (8) can be combined into one equation for the lateral displacement of the shell w as

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4} w_{,xxxx} + Nw_{,yy} + k(w - w_o) = 0 \tag{10}$$

Equation (4) can be integrated to obtain N_{yo} = Constant. The quantity N_{yo} represents the change in stress that occurs during buckling, and this constant may be taken equal to zero. Equation (5) can now be written as:

$$N\frac{d^2 w_o}{dy^2} + k(w - w_o) = 0 \tag{11}$$

Equations (10) and (11) can be solved exactly for the classical simple support boundary conditions ($w = w_o = 0$, $w_{,xx} = 0$; at $x = 0,L$) if the lateral displacements are chosen as $w = A \sin\frac{\pi x}{L}\sin\frac{ny}{r}$ and $w_o = B \sin\frac{\pi x}{L}\sin\frac{ny}{r}$. Substitution of these expressions for the lateral displacements (w and w_o) into equations (10) and (11) yields the following two homogeneous equations for A and B.

$$AD\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n}{r}\right)^2\right]^2 + \frac{A\frac{Et}{r^2}\left(\frac{\pi}{L}\right)^4}{\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n}{r}\right)^2\right]^2} - NA\left(\frac{n}{r}\right)^2 + k(A - B) = 0 \tag{12}$$

$$-NB\left(\frac{n}{r}\right)^2 + k(A - B) = 0 \tag{13}$$

By setting the determinant of the coefficients of A and B equal to zero, the following equation for the critical value of the stress resultant N can be obtained after some manipulation:

$$\frac{k_y{}^2\beta^2}{k_y\beta^2 + K} = \frac{\left(1 + \beta^2\right)^2}{\beta^2} + \frac{12Z^2}{\pi^4}\frac{1}{\left(1 + \beta^2\right)^2\beta^2} \tag{14}$$

where

$$Z = \frac{L^2}{rt}\sqrt{1 - \mu^2}; \quad K = \frac{kL^4}{D\pi^4}; \quad k_y = \frac{NL^2}{\pi^2 D}; \quad \beta = \frac{nL}{\pi r}$$

Equation (14) can be solved explicitly for k_y as:

$$k_y = \frac{k_{yp} + \sqrt{k_{yp}^2 + \frac{4k_{yp}K}{\beta^2}}}{2} \qquad (15)$$

where

$$k_{yp} = \frac{\left(1 + \beta^2\right)^2}{\beta^2} + \frac{12Z^2}{\pi^4}\,\frac{1}{\beta^2\left(1 + \beta^2\right)^2}$$

is the equation from which the buckling characteristics of a cylinder loaded by lateral pressure may be determined (see ref. 1).

RESULTS

The critical stress resultant is determined by minimizing k_y in equation (15) with respect to β. The results for several values of the spring stiffness parameter K are given in terms of dimensionless parameters in figure 2. For the limiting case in which the spring stiffness parameter K approaches zero the results are the same as those given in reference 1 for a cylinder loaded by lateral pressure. As the spring stiffness parameter approaches infinity, the buckling load becomes infinite. Note that no radial bond between the cylinder and springs is necessary to obtain this increase in the buckling load since the springs just prior to buckling are compressed.

For moderate Z and $K > 10^4$ it can be seen from figure 2 that a very good approximate formula for the buckling coefficient k_y can be written as:

$$k_y = \sqrt{K} \qquad (16)$$

or

$$N_{cr} = \sqrt{Dk} \tag{17}$$

The quantity N_{cr} is the critical tension load per unit length in the wrap.

DISCUSSION

The results of this analysis show that circular cylindrical shells loaded by pre-tensioned filament windings can have much higher classical buckling strengths than would be expected if the loading were replaced by an equivalent external lateral pressure. The analysis demonstrates that, for initially perfect shells, material compressibility in the thickness direction plays a major role in the mechanism through which these structures buckle. For configurations which are very stiff in the thickness direction, other effects such as initial imperfections might be important.

REFERENCE

1. Batdorf, S. B.: A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells. NACA Rep. 874, 1947.

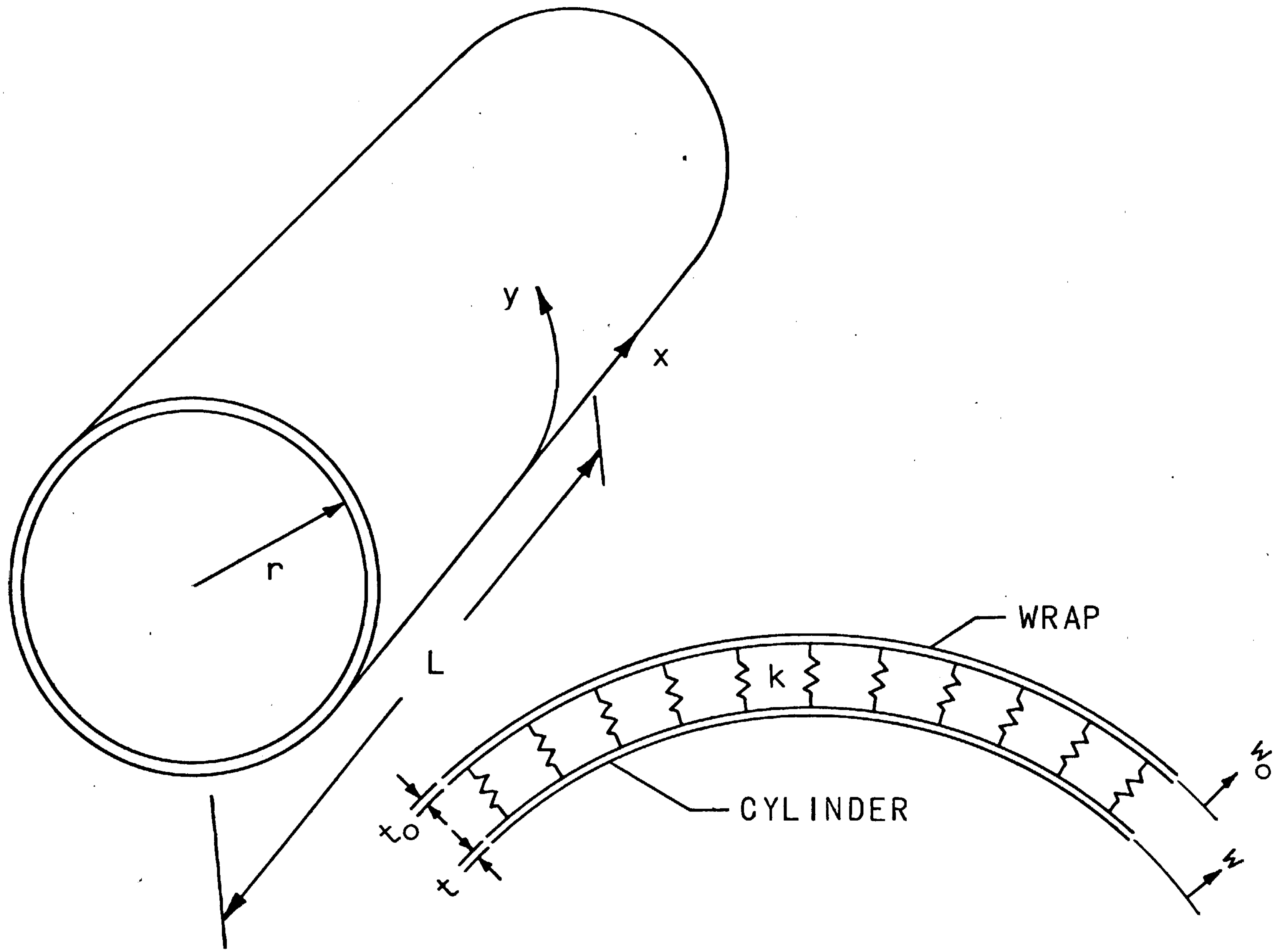

Fig. 1.- Configuration and coordinate system.

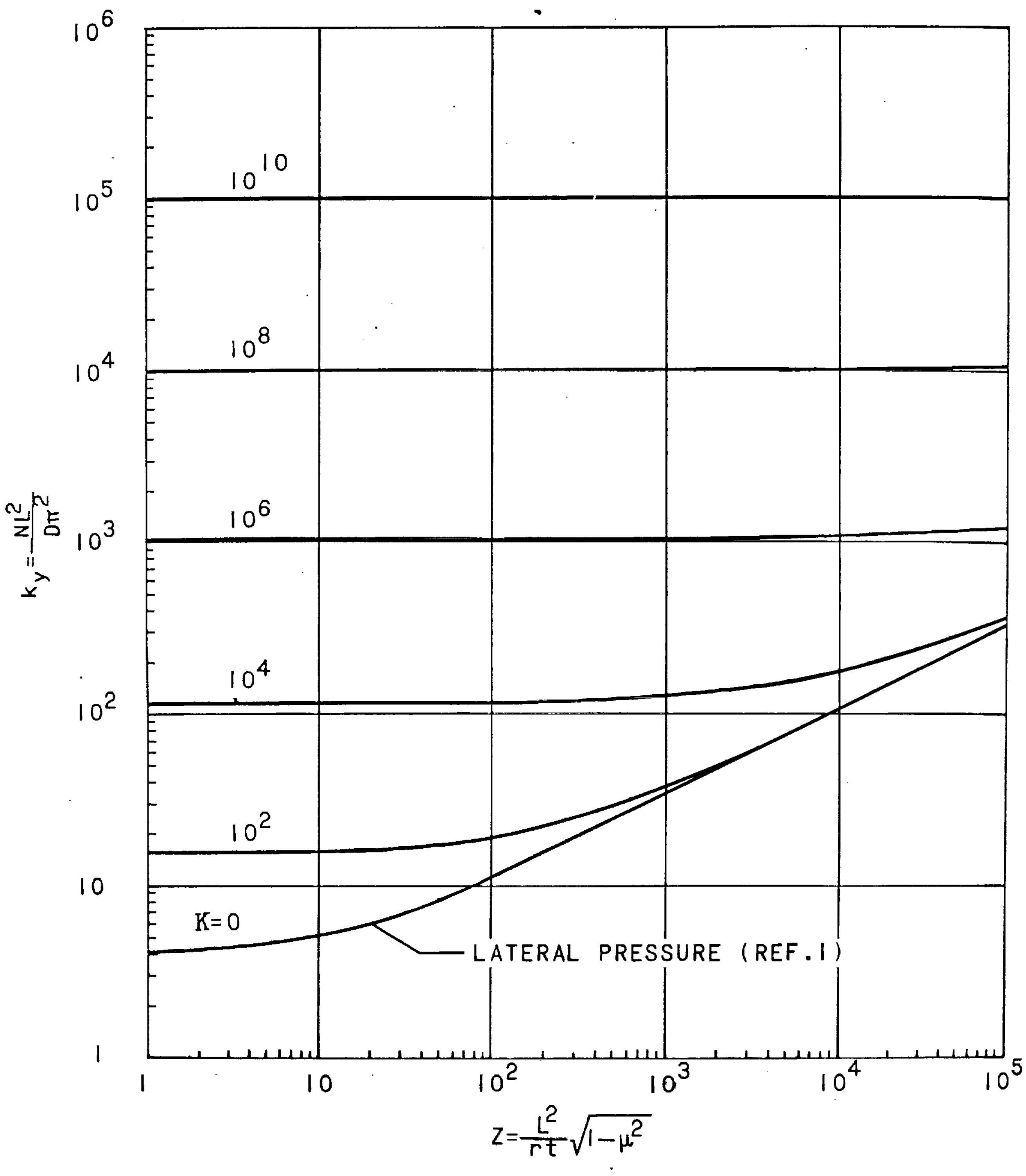

Fig. 2.- Buckling coefficient for cylinder wrapped with pretensioned filaments.